

The stable set polytope of $(P_6, \text{triangle})$ -free graphs and new facet-inducing graphs

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May 1, 2014

Abstract

The stable set polytope of a graph G , denoted as $\text{STAB}(G)$, is the convex hull of all the incidence vectors of stable sets of G . To describe a linear system which defines $\text{STAB}(G)$ seems to be a difficult task in the general case. In this paper we present a complete description of the stable set polytope of $(P_6, \text{triangle})$ -free graphs (and more generally of (P_6, paw) -free graphs). For that we combine different tools, in the context of a well known result of Chvátal [6] which allows to focus just on prime facet-inducing graphs, with particular reference to a structure result on prime $(P_6, \text{triangle})$ -free graphs due to Brandstädt et al. [4]. Also we point out some peculiarities of new facet-inducing graphs detected along this study with the help of a software.

Keywords: Stable set polytope; modular decomposition; facet-inducing graphs; P_6 -free graphs; triangle-free graphs.

1 Introduction

Let $G = (V, E)$ be a graph of n vertices. A *stable set* of G is a set of pairwise nonadjacent vertices of G . Each stable set S of G is characterized by its incidence vector, that is an n -vector whose i -th component is equal to 1 if vertex i is in S , and 0 otherwise. The *stable set polytope* of G , denoted by $\text{STAB}(G)$, is the convex hull of all the incidence vectors of stable sets of G .

A system $Ax \leq b$ of linear inequalities is called a *defining linear system* for the stable set polytope of a graph G if $\text{STAB}(G) = \{x \in \mathbb{R}^{|G|} : Ax \leq b\}$ holds and is *minimal* if all inequalities are facet-defining; the (sub)graphs supporting such facet-defining inequalities are called facet-inducing graphs. As shown by Padberg, there are two types of inequalities that are necessary for all graphs: the nonnegativity constraints $-x_v \leq 0$ for all vertices $v \in V$ and the clique constraints $\sum_{v \in Q} x_v \leq 1$ for all inclusion-wise maximal cliques $Q \subseteq G$. Those two types of inequalities yield a minimal defining linear system for $\text{STAB}(G)$ if and only if G is a perfect graph [6]. In particular a well known result due to Chudnovsky et al. [5] states that a graph is perfect if and only if it contains no induced odd holes and no induced odd co-holes.

Finding such a system for the stable set polytope of general graphs is a difficult task, see e.g. [15, 24, 25, 26]. That seems to be difficult also for graphs for which a maximum (weight)

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stable set can be computed in polynomial time: that is the case e.g. of claw-free graphs, though deep partial results have been stated, see e.g. [9, 11, 12, 14].

However, it is known (see [6]) how to construct the defining linear system for a graph obtained from substitution of a graph G_2 for a vertex v of a graph G_1 , provided that the defining linear system for $\text{STAB}(G_1)$ and for $\text{STAB}(G_2)$ are known. Furthermore, it is known (see e.g. [7, 19]) that every non-trivial homogeneous set of a facet-inducing graph induces a facet-inducing graph as well. Then for any graph class \mathcal{X} and for any $G \in \mathcal{X}$, an implicit description of $\text{STAB}(G)$ is given by the set $\mathcal{F}_P(\mathcal{X})$ of prime facet-inducing graphs of \mathcal{X} .

In this paper we determine the set $\mathcal{F}_P(\mathcal{X})$ for the class \mathcal{X} of $(P_6, \text{triangle})$ -free graphs and consequently give a complete description of $\text{STAB}(G)$ for every $G \in \mathcal{X}$ (these results are then extended to (P_6, paw) -free graphs). For that we combine, in analogy to the proof in [7, 19] for some extensions of P_4 -free graphs, different tools: Chvátal's results [6], properties of facet-inducing graphs due to Mahjoub [16], a structure result on bipartite P_6 -free graphs due to Fouquet, Giakoumakis, and Vanherpe [10], and intensively a structure result on non-bipartite prime $(P_6, \text{triangle})$ -free graphs due to Brandstädt, Klemmt, and Mahfud [4]. Also we point out some peculiarities of the new facet-inducing graphs detected along this study with the help of a software.

The description of the stable set polytope for subclasses of triangle-free graphs may be of additional interest for the following reasons: (i) for triangle-free graphs the Maximum Stable Set Problem remains NP-hard [23]; then the difficulty of describing their stable set polytope should be similar to that of the general case; (ii) for triangle-free graphs no substitution is possible (otherwise a triangle arises); then the difficulty of describing their stable set polytope is directly linked to that of detecting prime facet-inducing graphs.

2 Basic notation and preliminary

For any missing notation or reference let us refer to [3, 24].

Let $G = (V, E)$ be a graph with n vertices.

For any vertex-set $W \subseteq V$, let $N(W) = \{v \in V \setminus W : v \text{ is adjacent to some vertex of } W\}$; if $W = \{w\}$, then let us write $N(w)$ instead of $N(\{w\})$; in particular $|N(v)|$ is the *degree* of v . For any vertex-set $W \subseteq V$, let $G[W]$ denote the subgraph of G induced by W . For convenience, in some part of the paper let us write $G - W$ instead of $G[V \setminus W]$.

A *clique* of G is a set of pairwise adjacent vertices of G .

For $q > 1$: let K_q denote (the graph induced by) a clique with q vertices, let P_q denote an induced path with q vertices, and let C_q denote an induced cycle with q vertices. Graph K_3 is also called *triangle*. A *paw* has vertices a, b, c, d , and edges ab, ac, bc, ad , i.e., a paw is a one-vertex extension of a triangle.

Given a graph F , let us say that G is *F-free* if G contains no induced subgraph isomorphic to F . In particular, if G is both F_1 -free and F_2 -free for some graphs F_1, F_2 , then let us write G is (F_1, F_2) -free.

The *complement* of G , denoted by $\text{co-}G$, is the graph having the same vertices as G and where two vertices are adjacent in $\text{co-}G$ if and only if they are nonadjacent in G .

Two disjoint vertex-sets $X, Y \subseteq V$ have a *join* (a *co-join*) if each element of X is adjacent

(nonadjacent) to each element of Y . A vertex $v \in V$ *distinguishes* vertices $x, y \in V$ if $(v, x) \in E$ and $(v, y) \notin E$. A vertex set $M \subseteq V$ is a *module* of G if no vertex from $V \setminus M$ distinguishes two vertices from M . A module is trivial if it is either the empty set, a one-vertex or the entire vertex set V . Nontrivial modules are called *homogeneous sets*.

A graph is *prime* if it contains only trivial modules. The notion of modules is basic in the modular decomposition (or substitution) of graphs (see e.g. [18]). A homogeneous set M is *maximal* if no other homogeneous set properly contains M . It is well known that in a connected graph G with connected complement $\text{co-}G$, the maximal homogeneous sets are pairwise disjoint which means that every vertex is contained in at most one homogeneous set. The existence and uniqueness of the *modular decomposition tree* is based on this property, and linear time algorithms were designed to determine this tree (see e.g. [17]). The tree contains the vertices of the graph as its leaves, and the internal nodes are of three types: they represent a join or a co-join operation, or a prime graph.

2.1 STAB(G) and modular composition

Substitution of a graph F for a vertex v of a graph G consists of taking a disjoint union of F and $G - \{v\}$, and adding an edge between every vertex of F and every vertex of $G - \{v\}$ that was adjacent to v in G ; $G(v, F)$ denotes the graph obtained that way.

The following well known result of Chvátal (cf. Theorem 5.1 in [6]) gives the link between defining linear systems of $\text{STAB}(G)$ and modular composition of graphs: if one knows a defining linear system of $\text{STAB}(G)$ and a defining linear system of $\text{STAB}(F)$, then one knows a defining linear system of $\text{STAB}(G(v, F))$.

Theorem 1 ([6]) *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with $V_1 \cap V_2 = \emptyset$. For $k = \{1, 2\}$, let*

$$\begin{aligned} -x_u &\leq 0 & (u \in V_k) \\ \sum_{u \in V_k} a_{iu} x_u &\leq b_i & (i \in J_k) \end{aligned}$$

be a defining linear system of $\text{STAB}(G_k)$ (where J_k is its inequalities index-set). Let v be a node of G_1 and let G be the graph obtained from G_1 by substituting G_2 for v . For each $i \in J_1$, set $a_{iv}^+ = \max\{a_{iv}, 0\}$. Then

$$\begin{aligned} -x_u &\leq 0 & (u \in V_2 \cup (V_1 - v)) \\ a_{iv}^+ \sum_{u \in V_2} a_{ju} x_u + b_j \sum_{u \in V_1 - v} a_{iu} x_u &\leq b_i b_j & (i \in J_1, j \in J_2) \end{aligned}$$

is a defining linear system of $\text{STAB}(G)$. □

An inequality is *valid* for $\text{STAB}(G)$ if it is satisfied by each element of $\text{STAB}(G)$. A *face* of $\text{STAB}(G)$ is the set $\{\mathbf{x} \in \text{STAB}(G) : \sum_{i \in V} c_i x_i = b\}$ for some inequality $\sum_{i \in V} c_i x_i = b$ valid for G . A *facet* of $\text{STAB}(G)$ is a maximal proper face of $\text{STAB}(G)$. An inequality $\sum_{i \in V} c_i x_i \leq b$ is *facet-defining*, i.e., it belongs to a minimal defining linear system of $\text{STAB}(G)$, if and only if it is valid for $\text{STAB}(G)$ and $\{\mathbf{x} \in \text{STAB}(G) : \sum_{i \in V} c_i x_i = b\}$ is a facet of $\text{STAB}(G)$.

Each facet of $\text{STAB}(G)$ is uniquely determined up to multipliers (cf. Theorem 3.16 of [24]). In this sense, $\text{STAB}(G)$ admits a unique minimal defining linear system.

It is known that (see e.g. [24]): if $\{\mathbf{x} \in \text{STAB}(G) : \sum_{i \in V} c_i x_i = b\}$ is a facet of $\text{STAB}(G)$, then it is also a facet of $\text{STAB}(G[W])$, where $W = \{i \in V : c_i \neq 0\}$. Then let us focus on the following kind of graphs.

A graph $G = (V, E)$ of n vertices is *facet-inducing* if there exists a vector $\mathbf{c} = (c_1, \dots, c_n)^T$ with $c_i \neq 0$ for every i , and an integer b such that the inequality $\sum_{i \in V} c_i x_i \leq b$ is facet-defining for $\text{STAB}(G)$. Examples of facet-inducing graphs are the C_k and the $\text{co-}C_k$ for $k = 2j + 1$ and $j \geq 2$ [21, 22]. Let us call such an inequality $\sum_{i \in V} c_i x_i \leq b$ as a *full facet* of $\text{STAB}(G)$. Actually one can assume that $b = 1$, according to Theorem 4 of [7] which states that $b > 0$. Notice that $\text{STAB}(G)$ may have different full facets, see e.g. Figure 1 (c)-(d) of [2]. Then, for any facet-inducing graph G , let $\Phi(G)$ denote the set of full facets of $\text{STAB}(G)$.

Let us report the following result, one implication of which comes from Theorem 3.16 of [24].

Theorem 2 ([7]) *Let G be a graph of n vertices, $n > 1$, and let $\mathbf{c}^T \mathbf{x} \leq b$ be an inequality valid for $\text{STAB}(G)$, where $\mathbf{c} = (c_1, \dots, c_n)^T$ with $c_i \neq 0$ for every i . Then the following statements are equivalent:*

- (a) G is facet-inducing and $\mathbf{c}^T \mathbf{x} \leq b$ is facet-defining for $\text{STAB}(G)$;
- (b) there exists an $n \times n$ nonsingular matrix \mathbf{M} , whose rows are incidence vectors of n maximal stable sets of G , such that $\mathbf{M}\mathbf{c} = \mathbf{b}$, where \mathbf{b} denotes the vector whose components are all equal to b . \square

Note: As remarked above, one implication of Theorem 2 comes from Theorem 3.16 of [24]. Actually, also the other implication of Theorem 2 seems to be known earlier than [7], since it seems to be applied (as a known fact) in some argument given in [2].

For every graph class \mathcal{X} , let $\mathcal{F}(\mathcal{X})$ denote the class of all graphs in \mathcal{X} that are facet-inducing. Clearly one has:

Proposition 1 *For every graph class \mathcal{X} and for every $G \in \mathcal{X}$, if one knows $\mathcal{F}(\mathcal{X})$, then through $\{\Phi(H) : H \in \mathcal{F}(\mathcal{X})\}$ one knows (explicitly) a linear defining system of $\text{STAB}(G)$. \square*

Let $\mathcal{F}_{\mathcal{P}}(\mathcal{X})$ denote the class of all graphs in $\mathcal{F}(\mathcal{X})$ that are prime, and let $\mathcal{S}(\mathcal{F}_{\mathcal{P}}(\mathcal{X}))$ denote the class of all graphs that are obtained by possible repeated substitutions of any graph in $\mathcal{F}_{\mathcal{P}}(\mathcal{X})$ for a vertex of any graph in $\mathcal{F}_{\mathcal{P}}(\mathcal{X})$. Clearly, $\mathcal{F}_{\mathcal{P}}(\mathcal{X}) \subseteq \mathcal{F}(\mathcal{X}) \subseteq \mathcal{X}$ and $\mathcal{F}_{\mathcal{P}}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{F}_{\mathcal{P}}(\mathcal{X}))$. Note that, if one restricts attention to hereditary graph classes (i.e., defined by forbidding induced subgraphs), then $\mathcal{F}_{\mathcal{P}}(\mathcal{X})$ contains K_2 for every non-empty hereditary graph class \mathcal{X} . In particular, since $\mathcal{F}(\mathcal{X})$ is formed by cliques if and only if \mathcal{X} is a subclass of the class of perfect graphs [6], $\mathcal{F}_{\mathcal{P}}(\mathcal{X}) = \{K_2\}$ if and only if \mathcal{X} is a subclass of the class of perfect graphs.

By Theorem 2 one can prove the following proposition.

Proposition 2 ([19]) *Let G be a facet-inducing graph. Then every subgraph of G induced by a homogeneous set in G is facet-inducing. \square*

By Theorem 1 and Proposition 2 one obtains the following corollary, also quoted in [7, 19].

Corollary 1 *For every graph class \mathcal{X} , $\mathcal{F}(\mathcal{X}) = \mathcal{S}(\mathcal{F}_{\mathcal{P}}(\mathcal{X})) \cap \mathcal{X}$.* \square

Proof. By Theorem 1 one directly has $\mathcal{S}(\mathcal{F}_{\mathcal{P}}(\mathcal{X})) \cap \mathcal{X} \subseteq \mathcal{F}(\mathcal{X})$. Then let us prove that $\mathcal{F}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{F}_{\mathcal{P}}(\mathcal{X})) \cap \mathcal{X}$. Let $G \in \mathcal{F}(\mathcal{X})$. By Proposition 2 each node of the modular decomposition tree of G , different from those representing a join or a co-join operation, represents a prime facet-inducing graph. Then the assertion follows. \square

Then one has:

Proposition 3 *For every graph class \mathcal{X} and for every graph $G \in \mathcal{X}$, if one knows $\mathcal{F}_{\mathcal{P}}(\mathcal{X})$, then through $\{\Phi(H) : H \in \mathcal{F}_{\mathcal{P}}(\mathcal{X})\}$ one knows (implicitly) a defining linear system of $\text{STAB}(G)$.* \square

Proof. If one knows $\mathcal{F}_{\mathcal{P}}(\mathcal{X})$ and $\{\Phi(H) : H \in \mathcal{F}_{\mathcal{P}}(\mathcal{X})\}$, then one can get $\mathcal{F}(\mathcal{X})$ by Corollary 1 and $\{\Phi(H') : H' \in \mathcal{F}(\mathcal{X})\}$ by Theorem 1, that is a linear defining system of $\text{STAB}(G)$ up to nonnegativity constraints. \square

2.2 On the structure of facet-inducing graphs

Chvátal [6] and Mahjoub [16] proved several results on the structure of facet-inducing graphs. In the sequel let us report just those results which will be used later. Also let us report an observation from [7] and introduce two new observations.

First let us observe that every facet-inducing graph is connected: this fact, which is mentioned also in [2], can be derived by Corollary 1 and since every prime graph is connected.

The following lemma is an extract of the proof of Lemma 1 of [16] and is reported together with the proof, since such a proof idea/technique will be used later.

Lemma 1 ([16]) *Let G be a facet-inducing graph and $\mathbf{c}^T \mathbf{t} \leq 1$ be a full facet of $\text{STAB}(G)$. Let v be a vertex of G , of degree 2, with two non-adjacent neighbors a, b . Then $c_v \leq c_a$ and $c_v \leq c_b$.*

Proof. Let S^* be the family of all maximal stable sets of G whose incidence vectors \mathbf{t} enjoy $\mathbf{c}^T \mathbf{t} = 1$. Then the only equations satisfied by all the incidence vectors of members of S^* are positive multiples of $\mathbf{c}^T \mathbf{t} = 1$. We claim that there exists a stable set $S_0 \in S^*$ such that $a \in S_0$ and $b \notin S_0$. In fact, if this is not the case then for every stable set $S \in S^*$ the following holds: $a \in S$ implies $b \in S$; $a \notin S$ implies $|S \cap \{v, b\}| = 1$. Thus $t_v + t_b = 1$ holds for all the incidence vectors of members of S^* , a contradiction. Let $S'_0 = (S_0 \setminus \{a\}) \cup \{v\}$. Since S'_0 is also a stable set of G , we have $c_v \leq c_a$. Similarly, by symmetry, one obtains $c_v \leq c_b$. \square

A *cutset* of a graph $G = (V, E)$ is a subset W of V such that $G - W$ has more connected components than G . A *clique cutset* of G is a cutset of G which is also a clique of G .

Theorem 4.1 of [6] shows that, given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, if $(V_1 \cap V_2, E_1 \cap E_2)$ is a clique, then a defining linear system of $(V_1 \cup V_2, E_1 \cup E_2)$ is given by the union of linear defining systems of G_1 and G_2 respectively. Then by definition of facet-inducing graph one obtains:

Theorem 3 ([6]) *Every facet inducing-graph has no clique cutset.* \square

Let us say that a vertex u of a facet-inducing graph G with n vertices, with $n > 1$, is *critical* for G if there exists a matrix \mathbf{M} according to Theorem 2 such that the column of \mathbf{M} corresponding to vertex u has a unique entry equal to 1.

Observation 1 ([7]) *Let G be a facet-inducing graph. If v is a critical vertex for G , then $G - \{v\}$ is facet-inducing.* \square

Let us conclude this subsection by introducing two observations.

Let us say that a subgraph $G[H]$ of a graph G is *repeating* for G if for each maximal stable set S of G such that $S \cap H \neq \emptyset$ one has that $S \cap H$ is maximal for $G[H]$.

Observation 2 *Let G be a facet-inducing graph and $G[H]$ be a repeating subgraph of G . Then $G[H]$ contains $|H|$ maximal stable sets whose incidence vectors form a linearly independent set.*

Proof. Since G is facet-inducing, there exists a matrix \mathbf{M} according to Theorem 2. Since \mathbf{M} is nonsingular, the submatrix \mathbf{M}' of \mathbf{M} formed by the columns corresponding to the vertices of $G[H]$ has rank $|H|$. Then since $G[H]$ is repeating for G , the rows of \mathbf{M}' are incidence vectors of maximal stable sets of $G[H]$, and the assertion follows. \square

Observation 3 *Let G be a facet-inducing graph and $\mathbf{c}^T \mathbf{t} \leq 1$ be a full facet of $STAB(G)$. Let v be a vertex of G , of degree 2, with two non-adjacent neighbors a, b . Then there exists a maximal stable set S of G with incidence vector \mathbf{t}' such that $a, b \in S$, and $\mathbf{c}^T \mathbf{t}' = 1$.*

Proof. Let S^* be the family of all maximal stable sets of G whose incidence vectors \mathbf{t} are such that $\mathbf{c}^T \mathbf{t} = 1$. Then the only equations satisfied by all the incidence vectors of members of S^* are positive multiples of $\mathbf{c}^T \mathbf{t} = 1$. Assume to the contrary that there exists no maximal stable set in S^* containing both a and b . Then each maximal stable set in S^* contains exactly one vertex from $\{a, v, b\}$. Then $t_a + t_v + t_b = 1$ holds for all maximal stable sets of S^* , a contradiction. \square

3 Some observations on facet-inducing graphs which are either triangle-free or P_6 -free

Let us consider triangle-free graphs.

Observation 4 *Let \mathcal{X} be the class of triangle-free graphs. Then $\mathcal{F}(\mathcal{X}) = \mathcal{F}_{\mathcal{P}}(\mathcal{X}) = \mathcal{S}(\mathcal{F}_{\mathcal{P}}(\mathcal{X}))$.*

Proof. In fact every prime facet-inducing graph contains at least one edge; then to avoid a triangle one has $\mathcal{F}_{\mathcal{P}}(\mathcal{X}) = \mathcal{S}(\mathcal{F}_{\mathcal{P}}(\mathcal{X}))$; then by Corollary 1 the assertion follows. \square

Let us consider P_6 -free graphs.

Let us report a result due to Fouquet, Giakoumakis, and Vanherpe [10] (see also [13]). Given a bipartite graph $G = (S_1 \cup S_2, F)$, the *bi-complemented* graph \overline{G}^{bip} is the graph having the same vertex set $S_1 \cup S_2$ as G while its edge set is equal to $(S_1 \times S_2) \setminus F$.

Theorem 4 ([10]) *Let $G = (S_1 \cup S_2, F)$ be a connected bipartite P_6 -free graph. Then one of the following cases occurs:*

- (i) \overline{G}^{bip} is disconnected;
- (ii) there exist $S_1^* \subseteq S_1$ and $S_2^* \subseteq S_2$ such that $G[S_1^* \cup S_2^*]$ is complete bipartite, and $(S_1 \setminus S_1^*) \cup (S_2 \setminus S_2^*)$ is an independent set. \square

Lemma 2 *Let $G = (S_1 \cup S_2, F)$ be a connected bipartite P_6 -free graph, different to K_2 . Then G contains less than $|S_1 \cup S_2|$ maximal stable sets.*

Proof. For every subset U of $V = S_1 \cup S_2$, let $m(U)$ be the number of maximal stable sets contained in $G[U]$. Referring to Theorem 4, let us consider the following cases.

Assume that case (i) occurs. Then let K_1, \dots, K_t be the vertex sets of the connected components of \overline{G}^{bip} . Then $m(V) = m(K_1) + \dots + m(K_t) - 2t + 2$, where $2t$ is the number of the sides of each K_i ($i = 1, \dots, t$) and 2 is the number of the sides of V (which are S_1 and S_2). Thus since $t > 1$, if $m(K_i) \leq |K_i|$ for $i = 1, \dots, t$, then the lemma follows.

Assume that case (ii) occurs. If $S_1 \setminus S_1^* = S_2 \setminus S_2^* = \emptyset$, then $m(V) = 2$, i.e., the lemma follows since G is not a K_2 . If $S_1 \setminus S_1^* = \emptyset$ and $S_2 \setminus S_2^* \neq \emptyset$, then $m(V) = m(S_1^* \cup (S_2 \setminus S_2^*))$ (in fact there is just one maximal stable set of G containing vertices of S_2^* , that is S_2). If $S_1 \setminus S_1^* \neq \emptyset$ and $S_2 \setminus S_2^* = \emptyset$, then similarly $m(V) = m(S_2^* \cup (S_1 \setminus S_1^*))$. If $S_1 \setminus S_1^* \neq \emptyset$ and $S_2 \setminus S_2^* \neq \emptyset$, then similarly $m(V) = m(S_1^* \cup (S_2 \setminus S_2^*)) + m(S_2^* \cup (S_1 \setminus S_1^*)) - 1$ (since the maximal stable set $(S_1 \setminus S_1^*) \cup (S_2 \setminus S_2^*)$ has been considered twice). Thus, concerning the last three cases, if $m(S_1^* \cup (S_2 \setminus S_2^*)) \leq |S_1^* \cup (S_2 \setminus S_2^*)|$ and $m(S_2^* \cup (S_1 \setminus S_1^*)) \leq |S_2^* \cup (S_1 \setminus S_1^*)|$, then the lemma follows.

By Theorem 4 one can repeatedly iterate the above arguments for each K_i ($i = 1, \dots, t$), for $S_1^* \cup (S_2 \setminus S_2^*)$ and for $S_2^* \cup (S_1 \setminus S_1^*)$, until to reach subgraphs of G , say H , such that H either is empty or is an edge, that is such that $m(H) \leq |H|$. Then the lemma follows. \square

Definition 1 *A subgraph $G[H]$ with no isolated vertices of a graph G is a bi-module of G if:*

- (i) $G[H]$ is bipartite, i.e., $G[H] = (H_1 \cup H_2, F)$;
- (ii) H_1 (H_2) is a module of $G - H_2$ (of $G - H_1$).

According to the above definition, each edge of a graph G is trivially a bi-module of G .

Lemma 3 *Let G be a facet-inducing P_6 -free graph. Then each bi-module of G is an edge of G .*

Proof. Let $G[H]$ be a bi-module of G . By definition of bi-module and since $G[H]$ has no isolated vertices, $G[H]$ is a repeating subgraph of G . Then by Observation 2 $G[H]$ contains $|H|$ maximal stable sets whose incidence vectors form a linearly independent set.

If $G[H]$ is connected, then by Lemma 2 and since $G[H]$ is bipartite P_6 -free, $G[H]$ contains $|H|$ maximal stable sets whose incidence vectors form a linearly independent set if and only if $G[H]$ is an edge of G . If $G[H]$ is not connected, then, since each connected component of a bi-module is a bi-module as well, by the previous sentence $G[H]$ is formed by (at least two) disjoint edges: then $G[H]$ does not contain $|H|$ maximal stable sets whose incidence vectors form a linearly independent set, that is, this case is not possible. \square

Let us conclude this section by pointing out a class of $(P_6, \text{triangle})$ -free graphs which are not facet-inducing – this will be useful later.

A graph G is

- *matched co-bipartite* if G is partitionable into two cliques C_1, C_2 with $|C_1| = |C_2|$ or $|C_1| = |C_2| - 1$ such that the edges between C_1 and C_2 are a matching and at most one vertex in C_1 and C_2 is not covered by the matching;
- *co-matched bipartite* if it is the complement of a matched co-bipartite graph.

Notice that every co-matched bipartite graph is not facet-inducing, since it contains not enough maximal stable sets, according to Theorem 2.

Let us call *ferry* a graph $F = (X \cup Y \cup Z, E)$ where X, Y, Z are respectively stable sets, $X = \{x_0, x_1, \dots, x_m\}$, $Y = \{y_0, y_1, \dots, y_m\}$, x_0 may not exist and dominates Y , y_0 may not exist and dominates X , x_i is adjacent to each vertex of Y except for y_i for $i = 1, \dots, m$ (i.e., $X \cup Y$ induce a co-matching bipartite graph), $Z = \{z_1, \dots, z_l\}$ with $l \leq m$ and, for every $i = 1, \dots, l$, z_i is of degree 2 and is adjacent to x_i and y_i – see Figure 1.

Lemma 4 *Every ferry is not facet-inducing.*

Proof. Assume to the contrary that a ferry $F = (X \cup Y \cup Z, E)$ is facet-inducing. Let us prove the lemma only for the case in which x_0 and y_0 do not exist; the case in which they exist can be similarly treated. Then $Z \neq \emptyset$, since co-matched bipartite graphs are not facet-inducing. Since F is facet-inducing, let $\mathbf{c}^T \mathbf{t} \leq 1$ be a full facet of $\text{STAB}(F)$. Let S^* be the family of all maximal stable sets of F such that their incidence vectors \mathbf{t} enjoy $\mathbf{c}^T \mathbf{t} = 1$. Then the only equations satisfied by all the incidence vectors of members of S^* are positive multiplies of $\mathbf{c}^T \mathbf{t} = 1$. For brevity let us say that the members of S^* are *green* sets.

Let us observe that a maximal stable set of F may be just of three types: *side* if it is either X , or Y , or Z ; *cross* if it is formed by x_i, y_i and all z_j 's with $j \neq i$; *balance* if it is formed by a subset of m vertices picked up either in both X and Z (balance (X, Z)) or in both Y and Z (balance (Y, Z)).

For any vertex v of F , let us write $c(v)$ instead of c_v .

Claim 1 $c(z_i) \leq c(x_i)$ and $c(z_i) \leq c(y_i)$ for every $i = 1, \dots, m$

proof. It follows by Lemma 1. \square

Claim 2 X and Y are green

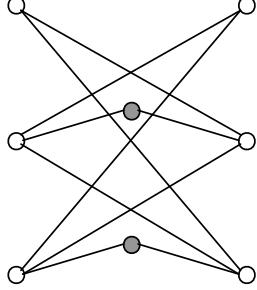


Figure 1: A ferry with $m = 3$ and $l = 2$ (without x_0 and y_0)

proof. Assume to the contrary that X is not green. Let $x_i \in X$ be such that z_i does exist. Notice that x_i is contained in at least one green balance (X, Z) : in fact, otherwise, x_i contained in at most one green maximal stable set (i.e., in a green cross); this implies that x_i is critical for F ; then by Observation 1, $F - \{x_i\}$ is facet-inducing; but $F - \{x_i\}$ contains a vertex of degree 1, i.e., vertex z_i , a contradiction to Theorem 3. Then let S be a green balance (X, Z) containing x_i . Let S_x (let S_z) denote the set of indices i such that $x_i \in S$ ($z_i \in S$). Then $\sum_{i \in S_x} c(x_i) + \sum_{j \in S_z} c(z_j) = 1$. By Claim 1 and since inequality $\mathbf{c}^T \mathbf{t} \leq 1$ is valid for $\text{STAB}(F)$, one has $c(x_j) = c(z_j)$ for every $j \in S_z$. Then $\sum_{i=1, \dots, m} c(x_i) = 1$, i.e., X is green. Similarly, by symmetry, one obtains that Y is green as well. \square

Claim 3 Z is not green

proof. Assume to the contrary that Z is green. Then $\sum_{i=1, \dots, m} c(z_i) = 1$. On the other hand, one has $c(x_1) + c(y_1) + \sum_{i=2, \dots, m} c(z_i) \leq 1$, since inequality $\mathbf{c}^T \mathbf{t} \leq 1$ is valid for $\text{STAB}(F)$. This is a contradiction since by Claim 1 $c(z_1) \leq c(x_1)$. \square

Claim 4 Each cross is green

proof. Assume to the contrary that the cross formed by x_i, y_i and all z_j 's with $j \neq i$ is not green. Then there exists no green maximal stable set containing both x_i and y_i . This contradicts Observation 3. \square

Let $\tilde{X} = \{x_i \in X : c(x_i) > c(z_i)\}$ and $\tilde{Y} = \{y_i \in Y : c(y_i) > c(z_i)\}$.

Claim 5 $|\tilde{X}| = 1$ and $|\tilde{Y}| = 1$

proof. Let us consider only \tilde{Y} . The case of \tilde{X} can be similarly treated by symmetry. Let us observe that $|\tilde{Y}| \geq 1$: in fact $|\tilde{Y}| = 0$ implies (by Claim 1) that Z is green, a contradiction to Claim 3.

Then assume by contradiction that $|\tilde{Y}| > 1$. Without loss of generality let $\tilde{Y} = \{y_1, \dots, y_q\}$ with $1 < q \leq m$. Let $G[H]$ be the graph induced by $\{x_1, \dots, x_q\} \cup \{z_1, \dots, z_q\}$. By definition

of \tilde{Y} , for each green maximal stable set S one has either $S \supseteq \tilde{Y}$ or $S \cap \tilde{Y} = \emptyset$. Then $G[H]$ is a repeating subgraph of G : then by Observation 2 $G[H]$ contains $|H|$ maximal stable sets whose incidence vectors form a linearly independent set. This is not possible since $G[H]$ is formed by (at least two) disjoint edges. \square

Let us conclude the proof of the lemma. By Claim 5, without loss of generality let $\tilde{X} = \{x_1\}$. By Claim 2, $\sum_{i=1,\dots,m} c(x_i) = 1$; then by definition of \tilde{X} , $c(x_1) + \sum_{i=2,\dots,m} c(z_i) = 1$. On the other hand by Claim 4, $c(x_1) + c(y_1) + \sum_{i=2,\dots,m} c(z_i) = 1$, a contradiction. \square

4 Structure of prime $(P_6, \text{triangle})$ -free graphs from [4]

In this section for the sake of completeness let us report those results from [4] which describe the structure of prime $(P_6, \text{triangle})$ -free graphs.

Throughout this section let $G = (V, E)$ be a non-bipartite prime $(P_6, \text{triangle})$ -free graph. For a subgraph H of G , a vertex not in H is a k -vertex of H (or for H) if it has exactly k neighbors in H . We say that H has no k -vertices if there is no k -vertex for H .

Since G is not bipartite, G must contain an odd cycle of length at least 5. In particular, since G is P_6 -free, G must contain a C_5 , say, C with vertices v_1, \dots, v_5 and edges $\{v_i, v_{i+1}\}, i \in \{1, \dots, 5\}$ (throughout this section, all index arithmetic with respect to a C_5 is done modulo 5). Obviously, in a triangle-free graph, a C_5 C has no 3-, 4- and 5-vertex, and 2-vertices of G are have nonconsecutive neighbors in C . Let X denote the set of 0-vertices of C , and for $i = 1, \dots, 5$, let Y_i denote the set of 1-vertices of C being adjacent to v_i , and let $Z_{i,i+2}$ denote the set of 2-vertices of C being adjacent to v_i and v_{i+2} .

Moreover, let $Y = Y_1 \cup \dots \cup Y_5$ and $Z = Z_{1,3} \cup Z_{2,4} \cup Z_{3,5} \cup Z_{4,1} \cup Z_{5,2}$. Obviously, $\{v_1, \dots, v_5\} \cup X \cup Y \cup Z$ is a partition of V .

The following result comes from Section 3 of [4].

Lemma 5 ([4]) *The following facts holds (for $i = 1, \dots, 5$):*

- (i) X is a stable set;
- (ii) X has a co-join to Y ;
- (iii) Y_i and $Z_{i,i+2}$ are stable sets;
- (iv) Y_i has a join to $Y_{i+2} \cup Y_{i+3}$, and a co-join to $Y_{i+1} \cup Y_{i+4}$;
- (v) $Z_{i,i+2}$ has a co-join to $Z_{i,i+3} \cup Z_{i+2,i+4}$;
- (vi) Y_i has a co-join to $Z_{i,i+2} \cup Z_{i,i+3}$;
- (vii) vertices in Y_i can only be distinguished by vertices in $Z_{i-1,i+1}$. \square

For the following we need the following notations:

$Z_{i,i+2}^0 := \{x : x \in Z_{i,i+2} \text{ and } x \text{ has a nonneighbor in } Z_{i-1,i+1} \text{ or in } Z_{i-1,i+1}\}$ for $i \in \{1, \dots, 5\}$, and let

$$Z_0 = \cup_{i=1}^5 Z_{i,i+2}^0.$$

Let X_0 denote the set of 0-vertices being adjacent to a vertex in Z_0 and let $G_0 := G[X_0 \cup Z_0]$.

The following result comes from Section 4 of [4].

Lemma 6 ([4]) *One of the following cases occurs: G_0 is*

- (i) *with no vertices;*
- (ii) *formed by at most five vertices, with $|Z_{i,i+2}^0| \leq 1$ for each $i = 1, \dots, 5$; in this case, $|X_0| \leq 1$ and X_0 has a join to Z_0 .*
- (ii) *a co-matched bipartite, namely $(Z_{i+1,i+3}^0 \cup Z_{i+2,i+4}^0, F')$; in this case each vertex of X_0 is adjacent to exactly a pair of nonadjacent vertices a, b with $a \in Z_{i+1,i+3}^0$ and $b \in Z_{i+2,i+4}^0$; in other words, $Z_{i+1,i+3}^0 \cup Z_{i+2,i+4}^0 \cup X_0$ induces a ferry;*
- (iv) *the disjoint union of two co-matched bipartite graphs, namely $(Z_{i+1,i+3}^0 \cup Z_{i+2,i+4}^0, F')$ and $(Z_{i,i+3}^0 \cup Z_{i+1,i+4}^0, F'')$; in this case, $X_0 = \emptyset$. \square*

Let $Z_{i,i+2}^1 := Z_{i,i+2} \setminus Z_{i,i+2}^0$ and $Z_1 := Z \setminus Z_0$. For $i \in \{1, \dots, 5\}$, let X_i denote the set of 0-vertices being adjacent to $Z_{i-1,i+1}^1$. Now, if for $i \in \{0, 1, \dots, 5\}$, X_i is trivial, we will omit the single vertex in X_i , i.e.: if X_i is nontrivial, then $X'_i = X_i$; if X_i is trivial, then $X'_i = \emptyset$.

For $i \in \{1, \dots, 5\}$, let $B_i := G[X'_i \cup Y_i \cup Z_{i-1,i+1}^1]$. By Lemma 5, $X \cup Y_i$ is a stable set, and thus, B_i is bipartite. Let X_T denote the union of trivial X_i , $i \in \{0, 1, \dots, 5\}$.

The *basic subgraphs* in G are the subgraphs G_0 and B_i , $i \in \{1, \dots, 5\}$.

Lemma 7 ([4])

The vertex sets X'_0, Z_0 of G_0 and the vertex sets $X'_i, Y'_i, Z_{i-1,i+1}^1$ of B_i , $i \in \{1, \dots, 5\}$, define a partition of $V \setminus (\{v_1, \dots, v_5\} \cup X_T)$. \square

Recall that by Lemma 5 (vii), vertices in Y_i can only be distinguished by vertices in $Z_{i-1,i+1}$. Thus, every vertex in $Z_{i-1,i+1}$ has either a join or a co-join to Y_{i+2} (Y_{i+3} , respectively).

Let $Z_{i-1,i+1;00}$ ($Z_{i-1,i+1;01}$, $Z_{i-1,i+1;10}$, $Z_{i-1,i+1;11}$) be the set of 2-vertices in $Z_{i-1,i+1}$ having a co-join to Y_{i+2} and Y_{i+3} (having a co-join to Y_{i+2} and a join to Y_{i+3} , having a join to Y_{i+2} and a co-join to Y_{i+3} , having a join to Y_{i+2} and a join to Y_{i+3}). Moreover, let $Z_{i-1,i+1;bc}^a = Z_{i-1,i+1}^a \cap Z_{i-1,i+1;bc}$, $a \in \{0, 1\}$, $bc \in \{00, 01, 10, 11\}$.

The *basic vertex subsets* of G are X'_0, X'_1, \dots, X'_5 , Y_1, \dots, Y_5 , and $Z_{i-1,i+1;bc}^a$, $a \in \{0, 1\}$, $bc \in \{00, 01, 10, 11\}$.

Theorem 5 (Structure Theorem [4])

For all pairs of basic vertex subsets U, W from different basic subgraphs, U has either a join or a co-join to W . \square

5 Structure of prime facet-inducing $(P_6, \text{triangle})$ -free graphs

In this section let us describe the structure of prime facet-inducing $(P_6, \text{triangle})$ -free graphs. Throughout this section let $G = (V, E)$ be a non-bipartite prime facet-inducing $(P_6, \text{triangle})$ -free graph. This is motivated from the fact that, since bipartite graphs are perfect, every (non-empty) bipartite prime facet-inducing graph is a K_2 . Then let us adopt the notation of Section 4.

Let us consider X'_i and Y_i .

Lemma 8 $X'_i = \emptyset$, for $i = 1, \dots, 5$.

Proof. Assume to the contrary that $X'_i \neq \emptyset$ for some $i = 1, \dots, 5$. For brevity let us write $T = Z_{i-1, i+1}^1$. Let $T' = T \cap N(X'_i)$: then $T' \neq \emptyset$, by definition of X'_i and since $X'_i \neq \emptyset$.

Claim 1 X'_i is a module of $G - T'$.

proof. Assume by contradiction that there is a vertex d of $G - T'$ distinguishing two vertices x_1, x_2 of X'_i . By Lemma 5 and since X'_i has a co-join to $Z_{i, i+2} \cup Z_{i, i+3}$ (otherwise a triangle arises with a vertex of T'), one has that $d \in Z_{i+1, i+3} \cup Z_{i+2, i+4}$, say $d \in Z_{i+1, i+3}$ being adjacent to x_1 and nonadjacent to x_2 , without loss of generality. If x_1 and x_2 share a neighbor t in T , then $x_2, t, x_1, d, v_{i+3}, v_{i+2}$ induce a P_6 . Otherwise, say x_1 is adjacent to $t_1 \in T$ and x_2 is adjacent to $t_2 \in T$, one has that $x_2, t_2, v_{i+4}, t_1, x_1, d$ induce a P_6 , contradiction. \square

Claim 2 T' has a join to $Y_{i+2} \cup Y_{i+3}$.

proof. In fact let $t \in T'$ be adjacent to $x \in X'_i$. Then t is adjacent to each vertex $y \in Y_{i+2}$, otherwise $x, t, v_{i+4}, v_{i+3}, v_{i+2}, y$ induce a P_6 . Similarly, by symmetry, t is adjacent to each vertex of Y_{i+3} . \square

To conclude the proof of the lemma let us consider the following cases.

Case 1 T has a co-join to Y_i

Then by Claims 1 and 2, T' and X'_i form a bi-module of G . By Lemma 3 they are an edge of G , i.e. $|X'_i| = 1$, a contradiction to the definition of X'_i .

Case 2 T has not a co-join to Y_i

This case means that $Y_i \neq \emptyset$. Then $Y_{i+2} \cup Y_{i+3} = \emptyset$: in fact otherwise by Lemma 5 and Claim 2 a triangle arises with a vertex of Y_i and a vertex of T' .

Let us prove that $T \cup \{v_i\}$ and $X'_i \cup Y_i$ form a bi-module of G . From one hand, $T \cup \{v_i\}$ is clearly a module of $G - (X'_i \cup Y_i)$. On the other hand, assume by contradiction that there is $d \in V \setminus (T \cup \{v_i\})$ distinguishing two vertices y, x of $X'_i \cup Y_i$. By Theorem 5, Claim 1 and since $Y_{i+2} \cup Y_{i+3} = \emptyset$, one has that $x \in X'_i$, $y \in Y_i$, $d \in Z_{i+1, i+3} \cup Z_{i+2, i+4}$, say $d \in Z_{i+1, i+3}$ without loss of generality. Let $t \in T$ be a neighbor of x . If d is adjacent to x and nonadjacent to y , then: if y is nonadjacent to t , then y, v_i, v_{i+4}, t, x, d induce a P_6 ; if y is adjacent to t , then v_i, y, t, x, d, v_{i+3} induce a P_6 . If d is adjacent to y and nonadjacent to x , then: if y is nonadjacent to t , then $y, d, v_{i+3}, v_{i+4}, t, x$ induce a P_6 ; if y is adjacent to t , then $v_{i+2}, v_{i+3}, d, y, t, x$ induce a P_6 , a contradiction.

Then $T \cup \{v_i\}$ and $X'_i \cup Y_i$ form a bi-module of G . By Lemma 3 they are an edge of G , i.e., $T = \emptyset$ and consequently $X'_i = \emptyset$, a contradiction. \square

Lemma 9 B_i is a stable set, for $i = 1, \dots, 5$.

Proof. Assume to the contrary that B_i is not a stable set for some $i = 1, \dots, 5$. For brevity let us write $T = Z_{i-1,i+1}^1$. By Lemma 8, $X'_i = \emptyset$: then B_i is formed by T and Y_i .

First assume that the possible trivial X_i does not exist. Then by Theorem 5, $T \cup \{v_i\}$ and Y_i form a bi-module of G . Then by Lemma 3, they are an edge of G (i.e., $T = \emptyset$), which implies that B_i is a stable set, a contradiction.

Then assume that the possible trivial X_i does exist, say vertex x . Since G is triangle-free, x is adjacent to no vertex of $Z_{i,i+2} \cup Z_{i,i+3}$. Then one can apply an argument similar to that of Lemma 8, with $\{x\}$ instead of X'_i , to get a contradiction (notice that Case 1 is not possible since B_i is not a stable set). \square

Lemma 10 The following facts hold for $i = 1, \dots, 5$:

- (i) $Z_{i-1,i+1}$ has a co-join to Y_i ;
- (ii) $|Y_i| \leq 1$.

Proof. Proof of (i). It follows by Theorem 5 and by Lemma 9. Proof of (ii). It follows by Theorem 5, by (i) and since G is prime. \square

Remark 1. According to Lemma 10 (ii), throughout the remaining part of the paper let us denote as y_i the possible vertex of Y_i for $i = 1, \dots, 5$.

Lemma 11 $Z_{i-1,i+1}$ has a join to $\{y_{i+2}, y_{i+3}\}$, for $i = 1, \dots, 5$.

Proof. Let $z \in Z_{i-1,i+1}$.

First assume that y_i does exist. Then by Lemma 5, y_i is adjacent to y_{i+2}, y_{i+3} . Then z is adjacent to y_{i+2} , otherwise $v_{i+4}, z, v_{i+1}, v_{i+2}, y_{i+2}, y_i$ induce a P_6 . Similarly by symmetry z is adjacent to y_{i+3} .

Then assume that y_i does not exist and that there is $z^0 \in Z_{i,i+2} \cup Z_{i,i+3}$ nonadjacent to z , say $z^0 \in Z_{i,i+2}$ without loss of generality by symmetry. Then y_{i+2} is adjacent to z , otherwise $y_{i+2}, v_{i+2}, z^0, v_i, v_{i+4}, z$ induce a P_6 . Then let us consider y_{i+3} . Assume by contradiction that z is nonadjacent to y_{i+3} . By Lemma 6, z is adjacent to each vertex of $Z_{i,i+2} \setminus \{z^0\}$. Then to avoid that $\{v_i, z\}$ and $\{z^0\}$ form a bi-module of G , either z is adjacent to a vertex $z' \in Z_{i,i+3}$ nonadjacent to z^0 , that is $z, v_{i+1}, v_i, z', v_{i+3}, y_{i+3}$ induce a P_6 , or z is adjacent to a vertex x of X_T , that is $x, z, v_{i+1}, v_{i+2}, v_{i+3}, y_{i+3}$ induce a P_6 , a contradiction.

Finally assume that y_i does not exist and that z dominates $Z_{i,i+2} \cup Z_{i,i+3}$. Then to avoid that $\{v_i, z\}$ forms a module of G , z is adjacent to at least one vertex q from $X_T \cup \{y_{i+2}, y_{i+3}\}$, and to avoid that $\{v_i, z\}$ and $\{q\}$ form a bi-module of G , z is adjacent to at least two vertices from $X_T \cup \{y_{i+2}, y_{i+3}\}$ (recall that z is adjacent to at most one vertex of X_T). Then z is adjacent to at least one vertex from $\{y_{i+2}, y_{i+3}\}$, say y_{i+2} without loss of generality by symmetry. Moreover, if y_{i+3} does exist, then z is adjacent to y_{i+3} as well, otherwise $x \in X_T$, $z, v_{i+1}, v_{i+2}, v_{i+3}, y_{i+3}$ induce a P_6 . \square

Let us consider X_T . Recall that each vertex of Z is adjacent to at most one vertex of X_T .

Lemma 12 *If $x \in X_T$ is adjacent to $z \in Z_{i-1,i+1}^1$, then one of the following cases occurs:*

- (i) $X_T = \{x\}$ and $Z = \{z, \bar{z}\}$ with $\bar{z} \in Z_{i+1,i+3}$;
- (ii) $X_T = \{x\}$ and $Z = \{z, \bar{z}, \tilde{z}\}$ with $\bar{z} \in Z_{i+1,i+3}$, $\tilde{z} \in Z_{i+2,i+4}$, z, \bar{z}, \tilde{z} mutually nonadjacent, and x adjacent to each of them.

Proof. The proof is given by the following claims.

Claim 1 x is adjacent to a vertex $\bar{z} \in Z_{i+1,i+3} \cup Z_{i+2,i+4}$, say $\bar{z} \in Z_{i+1,i+3}$

proof. Let us observe that x is adjacent to no vertex $a \in Z_{i-1,i+1}^1 \setminus \{z\}$, otherwise $\{a, z\}$ forms a module of G . Then since z can not have degree 1 (by Theorem 3), $z \in Z_{i-1,i+1}^1$ and G is triangle-free, the claim follows. \square

Claim 2 $Z_{i-1,i+1}^1 = \{z\}$

proof. Assume to the contrary that there is $z' \in Z_{i-1,i+1}^1$ different to z .

If $\{y_{i+2}, y_{i+3}\} = \emptyset$, then to avoid that either $\{v_i, z'\}$ forms a module of G or $\{v_i, z'\}$ and $\{y_i\}$ form a bi-module of G , z' is adjacent to x ; then $\{z, z'\}$ forms a module of G , a contradiction.

Then assume that $\{y_{i+2}, y_{i+3}\} \neq \emptyset$. By Lemma 11 z and z' are adjacent to y_{i+2}, y_{i+3} . Then to avoid that $\{z, z'\}$ is a module of G , z' is nonadjacent to x . Notice that y_{i+2} does not exist, otherwise by Lemma 10 $x, \bar{z}, v_{i+3}, v_{i+4}, z', y_{i+2}$ induce a P_6 . Then there exists y_i , otherwise $\{v_i, z'\}$ and $\{y_{i+3}\}$ form a bi-module of G . Then $x, \bar{z}, y_i, y_{i+3}, z', v_{i+4}$ induce a P_6 , a contradiction. \square

Claim 3 x is adjacent to no vertex of $Z_{i-1,i+1}^0$

proof. Assume to the contrary that x is adjacent to $z^0 \in Z_{i-1,i+1}^0$. To avoid that $\{z, z^0\}$ forms a module of G , there is $q \in Z_{i,i+2} \cup Z_{i,i+3}$, say without loss of generality $q \in Z_{i,i+3}$, adjacent to z and nonadjacent to z^0 . Then x is adjacent to q , otherwise $z^0, x, z, q, v_{i+3}, v_{i+2}$ induce a P_6 . Then x, q, z induce a triangle, a contradiction. \square

Claim 4 If $\bar{z} \in Z_{i+1,i+3}^1$, then case (i) occurs.

proof. Let us show that $Z_0 = \emptyset$. Assume by contradiction that there is $z^0 \in Z_0$ and consider the following occurrences which are exhaustive by symmetry. If $z^0 \in Z_{i,i+2}^0$, then by definition of Z^0 there is $q \in Z_{i-1,i+1}^0$ (without loss of generality, by symmetry) nonadjacent to z^0 ; then $q, v_{i-1}, v_i, z^0, \bar{z}, x$ induce a P_6 , a contradiction. If $z^0 \in Z_{i-1,i+1}^0$, then by definition of Z^0 and by the previous fact there is $q' \in Z_{i,i+3}^0$ nonadjacent to z^0 : then y_{i+2} does not exist, otherwise by Lemma 11 q', z, y_{i+2} induce a triangle; also y_i does not exist, otherwise $x, \bar{z}, y_i, v_i, v_{i+4}, z^0$ induce a P_6 ; then to avoid that $\{v_i, z\}$ and $\{x\}$ form a bi-module of G , y_{i+3} does exist (adjacent to z); also to avoid that $\{v_{i+2}, \bar{z}\}$ and $\{x\}$ form a bi-module of G , y_{i+4} does exist (adjacent to \bar{z}); then $z^0, y_{i+3}, z, x, \bar{z}, y_{i+4}$ induce a P_6 , a contradiction. If $z^0 \in Z_{i,i+3}^0$, then by definition of Z_0 and by the previous fact there is $q'' \in Z_{i+2,i+4}^0$ nonadjacent to z^0 ; then $v_i, z^0, z, x, \bar{z}, q''$ induce a P_6 . Then the assertion follows.

Let us show that $Z_1 = \{z, \bar{z}\}$. By Claim 2 and its symmetric version, $Z_{i-1,i+1}^1 = \{z\}$ and $Z_{i+1,i+3}^1 = \{\bar{z}\}$. Consider $Z_{i,i+2}^1$. Assume by contradiction that there is $q \in Z_{i,i+2}^1$. If y_{i+1} does exist, then: if q is adjacent to a vertex x' of X_T ($x' \neq x$ to avoid a triangle), then by the symmetric version of Claim 1 and since $Z_0 = \emptyset$, there is $q' \in Z_{i+2,i+4}^1$ (w.l.o.g.) adjacent to x' (and to y_{i+1} , by Lemma 11), and consequently x, z, q, x', q', y_{i+1} induce a P_6 ; if q is adjacent

to no vertex of X_T , then to avoid a bi-module of G , q is adjacent either to y_{i+3} (thus q, y_{i+3}, z induce a triangle) or to y_{i+4} (thus q, y_{i+4}, \bar{z} induce a triangle). If y_{i+1} does not exist, then to avoid a bi-module of G , q is adjacent either to y_{i+3} or to y_{i+4} , a contradiction similar to the previous sentence. Then $Z_{i,i+2}^1 = \emptyset$. Consider $Z_{i,i+3}^1$. Assume by contradiction that there is $p \in Z_{i,i+3}^1$. Then p is adjacent to no vertex x' of X_T ($x' \neq x$ to avoid a triangle), otherwise $x', p, v_i, v_{i+1}, \bar{z}, x$ induce a P_6 . Also, y_{i+4} does not exist, otherwise $v_i, p, z, x, \bar{z}, y_{i+4}$ induce a P_6 . Then to avoid a bi-module of G , p is adjacent to y_{i+1}, y_{i+2} (i.e., they exist), and consequently by Lemma 11 p, y_{i+2}, z induce a triangle, a contradiction. Then $Z_{i,i+3}^1 = \emptyset$. Similarly, by symmetry, $Z_{i+2,i+4}^1 = \emptyset$. Then the assertion follows. \square

Claim 5 *If $\bar{z} \in Z_{i+1,i+3}^0$, then case (ii) occurs.*

proof. Let us show that $Z_0 = \{\bar{z}, \tilde{z}\}$, where \tilde{z} is a vertex of $Z_{i+2,i+4}^0$. First let us observe that: $Z_{i-1,i+1}^0 = \emptyset$, otherwise by Lemma 6 x should be adjacent to a vertex of $Z_{i-1,i+1}^0 = \emptyset$, a contradiction to Claim 3; $Z_{i,i+2}^0 = Z_{i,i+3}^0 = \emptyset$, otherwise by Lemma 6 x should be adjacent to a vertex of such sets, but then a triangle arises since $z \in Z_{i-1,i+1}^1$. Then by definition of Z_0 and by Lemma 6 there exists $\tilde{z} \in Z_{i+2,i+4}^0$, with \tilde{z} adjacent to x by Lemma 6. If y_i does exist, then $(Z_{i+1,i+3}^0 \setminus \{\bar{z}\}) \cup (Z_{i+2,i+4}^0 \setminus \{\tilde{z}\}) = \emptyset$, otherwise since by Lemma 6 \tilde{z} (\bar{z}) is adjacent to each vertex of $Z_{i+1,i+3}^0 \setminus \{\bar{z}\}$ (of $Z_{i+2,i+4}^0 \setminus \{\tilde{z}\}$), Lemma 11 imply that G has a triangle involving \tilde{z} or \bar{z} respectively. If y_i does not exist, then to avoid a bi-module of G involving z , at least one from $\{y_{i+2}, y_{i+3}\}$ does exist, say y_{i+2} without loss of generality; then $Z_{i+1,i+3}^0 \setminus \{\bar{z}\} = \emptyset$ (otherwise y_{i+2}, z, x, \bar{z} , a vertex of $Z_{i+1,i+3}^0$ and v_{i+3} induce a P_6), and consequently $Z_{i+2,i+4}^0 \setminus \{\tilde{z}\} = \emptyset$ (by definition of Z_0 and by Lemma 6). Then the assertion follows.

Let us show that $Z_1 = \{z\}$. By Claim 2, $Z_{i-1,i+1}^1 = \{z\}$. Notice that $Z_{i,i+2}^1 = \emptyset$: in fact if there is $q \in Z_{i,i+2}^1$, then q is nonadjacent to x (to avoid a triangle) and in general to no vertex $x' \in X_T$ (to avoid that $x', q, \bar{z}, x, \tilde{z}, v_{i+4}$ induce a P_6); then to avoid a bi-module of G , q is adjacent either to y_{i+3} (and thus q, y_{i+3}, z form a triangle) or to y_{i+4} (and thus q, y_{i+3}, \bar{z} form a triangle). Similarly by symmetry, $Z_{i,i+2}^1 = \emptyset$. Furthermore, $Z_{i+1,i+3}^1 = \emptyset$: in fact if there is $p \in Z_{i+1,i+3}^1$, then p is adjacent to x (to avoid a triangle) and in general to no vertex $x' \in X_T$ (to avoid that $x', p, v_{i+3}, \bar{z}, x, z$ induce a P_6); then to avoid a bi-module of G , p is adjacent either to y_i (and thus p, y_i, \tilde{z} form a triangle) or to y_{i+4} (this means, by the previous facts, that y_{i+2} does exist to avoid that $\{v_{i+2}, p\}$ and $\{y_{i+4}\}$ form a bi-module, and consequently that $x, z, y_{i+2}, y_{i+4}, p, v_{i+3}$ induce a P_6). Similarly, by symmetry, $Z_{i+2,i+4}^0 = \emptyset$. Then the assertion follows. \square

Then the lemma is proved. \square

Let us consider Z and subgraph G_0 .

Lemma 13 *The following facts hold for $i = 1, \dots, 5$:*

- (i) $|Z_{i-1,i+1}^1| \leq 1$;
- (ii) *if there is an edge between $Z_{i+1,i+3}$ and $Z_{i+2,i+4}$, then $Y_i = Z_{i-1,i+1}^1 = \emptyset$;*
- (iii) *if $Z_{i+1,i+3}^0 \neq \emptyset$ and $Z_{i+2,i+4}^0 \neq \emptyset$, then $Z_{i,i+2}^1 = Z_{i,i+3}^1 = \emptyset$;*
- (iv) *if $Z_{i+1,i+3}^0 \neq \emptyset$ and $Z_{i+2,i+4}^0 \neq \emptyset$ and $Z_{i,i+3}^0 \neq \emptyset$, then $Z_1 = \emptyset$.*

Proof. Proof of (i). It follows by Theorem 5, Lemmas 11 and 12, and since G is prime.

Proof of (ii). Let $a \in Z_{i+1,i+3}$ be adjacent to $b \in Z_{i+2,i+4}$. By Lemma 11 and since G is triangle-free, one has $Y_i = \emptyset$. Let us consider $Z_{i-1,i+1}^1$. Assume by contradiction that there is $z \in Z_{i-1,i+1}^1$. By Lemma 12, z is adjacent to no vertex of X_T . Then, since $Y_i = \emptyset$, to avoid that either $\{v_i, z\}$ form either a module of G or $\{v_i, z\}$ and one vertex from $\{y_{i+2}, y_{i+3}\}$ form either a module of G , both y_{i+2} and y_{i+3} do exist. Then $y_{i+2}, z, y_{i+3}, v_{i+3}, a, b$ induce a P_6 , a contradiction.

Proof of (iii). First let us observe that there exist $a \in Z_{i+1,i+3}^0$ and $b \in Z_{i+2,i+4}^0$ mutually nonadjacent: in fact otherwise, by definition of Z^0 , there are $a' \in Z_{i,i+2}$ nonadjacent to a , and $b' \in Z_{i,i+3}$ nonadjacent to b , with $a', v_i, b', v_{i+3}, a, b$ induce a P_6 .

Let us consider $Z_{i,i+2}^1$. Assume by contradiction that there is $z \in Z_{i,i+2}^1$. Then: z is adjacent to no vertex of X_T , by Lemma 12; y_{i+4} does not exist, otherwise by Lemma 11 z, a, y_{i+4} induce a triangle. If y_{i+1} does not exist, then either $\{v_{i+1}, z\}$ forms a module of G or $\{v_{i+1}, z\}$ and $\{y_{i+3}\}$ form a bi-module of G . If y_{i+1} does exist, then y_{i+3} does exist too, otherwise $\{v_{i+1}, z\}$ and $\{y_{i+1}\}$ form a bi-module of G : then $a, z, y_{i+3}, y_{i+1}, b, v_{i+4}$ induce a P_6 , a contradiction. Then $Z_{i,i+2}^1 = \emptyset$. Similarly by symmetry $Z_{i,i+3}^1 = \emptyset$.

Proof of (iv). By (iii), $Z_1 \setminus Z_{i+2,i+4}^1 = \emptyset$. Assume by contradiction that there is $z \in Z_{i+2,i+4}^1$. Then to avoid that either $\{v_{i+3}, z\}$ forms a module of G or $\{v_{i+3}, z\}$ and $\{y_{i+3}\}$ form a bi-module of G , there exists at least one vertex from $\{y_i, y_{i+1}\}$: then a triangle arises by Lemma 11. \square

Remark 2. According to Lemma 13 (i), throughout the remaining part of the paper let us denote as $z_{i-1,i+1}$ the possible vertex of $Z_{i-1,i+1}^1$ for $i = 1, \dots, 5$.

Lemma 14 *Case (iii) of Lemma 6 can not occur.*

Proof. Let $Z_{i+1,i+3}^0 \cup Z_{i+2,i+4}^0$ induce the co-matched bipartite graph forming $G[Z_0]$. In particular, one can assume that there is at least one edge from $Z_{i+1,i+3}^0$ to $Z_{i+2,i+4}^0$ (otherwise case (ii) of Lemma 6 occurs). Then by Lemma 13 (ii), y_i does not exist, and by Lemma 13 (iii), $Z_1 \setminus (Z_{i+1,i+3}^1 \cup Z_{i+2,i+4}^1) = \emptyset$. Then in Z_1 there are at most $z_{i+1,i+3}$ and $z_{i+2,i+3}$. Then: if $X_0 = \emptyset$, then $Z_{i+1,i+3}^0$ and $Z_{i+2,i+4}^0$ form a bi-module of G , which is not possible by Lemma 3; if $X_0 \neq \emptyset$, then one can check that G is ferry, which is not possible by Lemma 4. \square

Lemma 15 *Case (iv) of Lemma 6 can not occur.*

Proof. Let $Z_{i+1,i+3}^0 \cup Z_{i+2,i+4}^0$ induce one of the two co-matched bipartite graph forming $G[Z_0]$. In particular, one can assume that there is at least one edge from $Z_{i+1,i+3}^0$ to $Z_{i+2,i+4}^0$ (otherwise case (ii) of Lemma 6 occurs). Then, since $X_0 = \emptyset$, one can prove the lemma by applying the argument of Lemma 14. \square

Then let us summarize the above results by the following theorem.

Theorem 6 *Every prime facet-inducing $(P_6, \text{triangle})$ -free graph is a subgraph of one the graphs H_1, H_2, H_3 drawn respectively in Figures 2, 3, 4.*

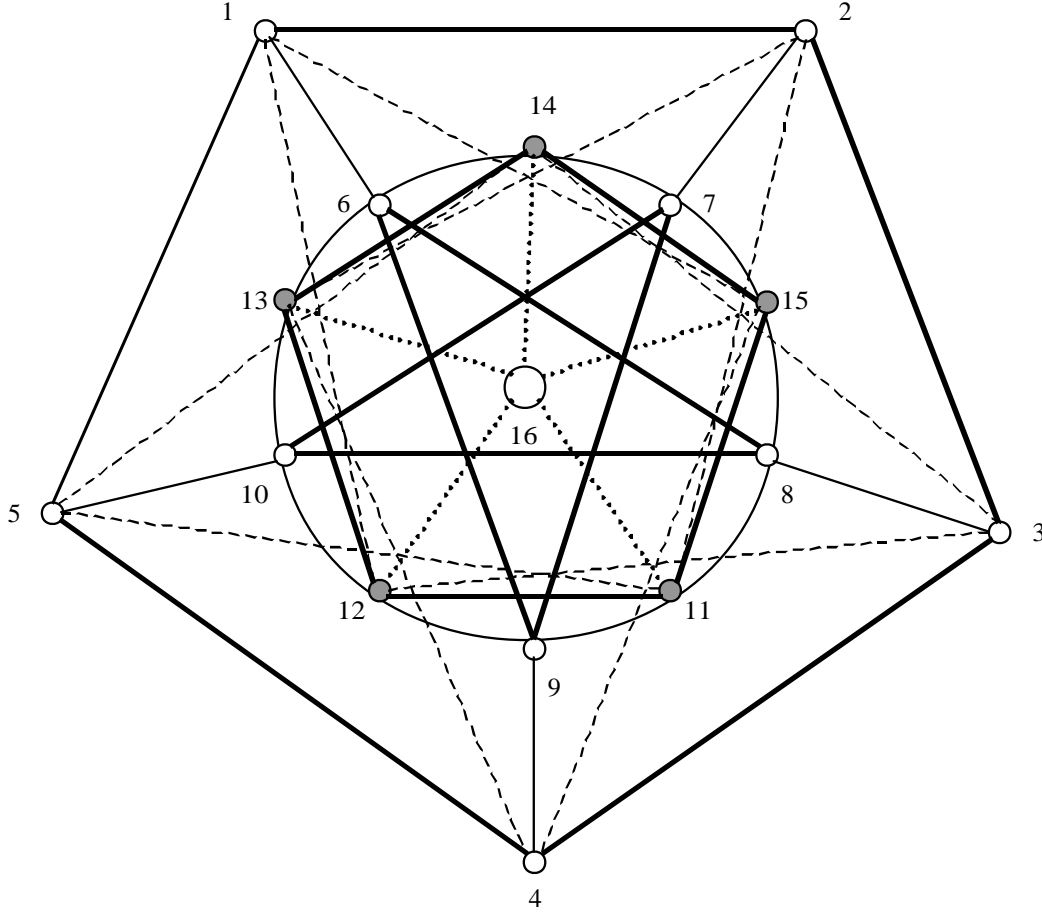


Figure 2: The graph H_1

Proof. Let G be a (non-empty) prime facet-inducing $(P_6, \text{triangle})$ -free graph. If G is bipartite, then G is perfect, i.e., $G = K_2$. Then assume that G is not bipartite. By Lemmas 14 and 15, only cases (i) and (ii) of Lemma 6 may occur. By the other results of this section, G has at most 21 vertices. In fact, one has $|Y_i| \leq 1, |Z_i^1| \leq 1, |Z_i^0| \leq 1$ for $i = 1, \dots, 5$; furthermore, $X'_i = \emptyset$, and $|X_T| \leq 1$ (since: if $x \in X_T$ is adjacent to a vertex of Z_1 , then Lemma 12 holds; otherwise, Lemma 6 (i)-(ii) holds).

Then let us focus on the following cases.

Case 1 Case (i) of Lemma 6 occurs.

This means that $Z_0 = \emptyset$. Then G is a subgraph of the graph H_1 where the vertices of Z are gray and the vertex of X_T is the big central one.

Case 2 Case (ii) of Lemma 6 occurs.

Subcase 2.1 $Z_1 = \emptyset$.

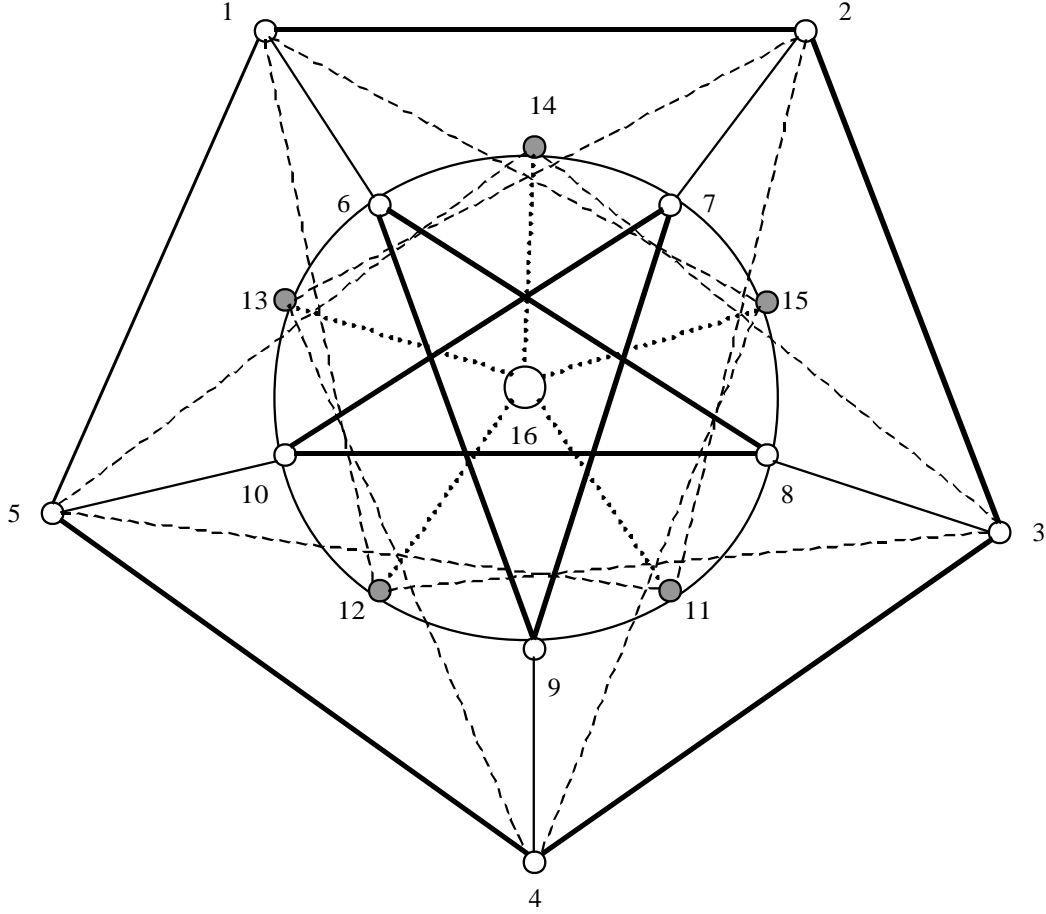


Figure 3: The graph H_2

Then G is a subgraph of the graph H_2 where the vertices of Z are gray and the vertex of X_T is the big central one.

Subcase 2.2 $Z_1 \neq \emptyset$ and the vertex of X_T is adjacent to a vertex of Z_1 .

Then by Lemma 12, G is a subgraph of the graph H_2 where the vertices of Z are gray and the vertex of X_T is the big central one.

Subcase 2.3 $Z_1 \neq \emptyset$ and the vertex of X_T is adjacent to no vertex of Z_1 .

Then by Lemma 11 and by definition of Z_0 , there are exactly two consecutive vertices of Z_0 , say $z_{2,4}^0, z_{3,5}^0$, plus possibly: y_2, y_3, y_4, y_5 (by Lemma 11 y_1 does not exist), $z_{2,4}, z_{3,5}, z_{2,5}$, and a vertex $x \in X_0$ adjacent only to $z_{2,4}^0, z_{3,5}^0$. Then G is a subgraph of the graph H_3 where the vertices of Z are gray and the vertex of X_T is the big central one. \square

Let us observe that H_1 is not $(P_6, \text{triangle})$ -free, while H_2 and H_3 are $(P_6, \text{triangle})$ -free.

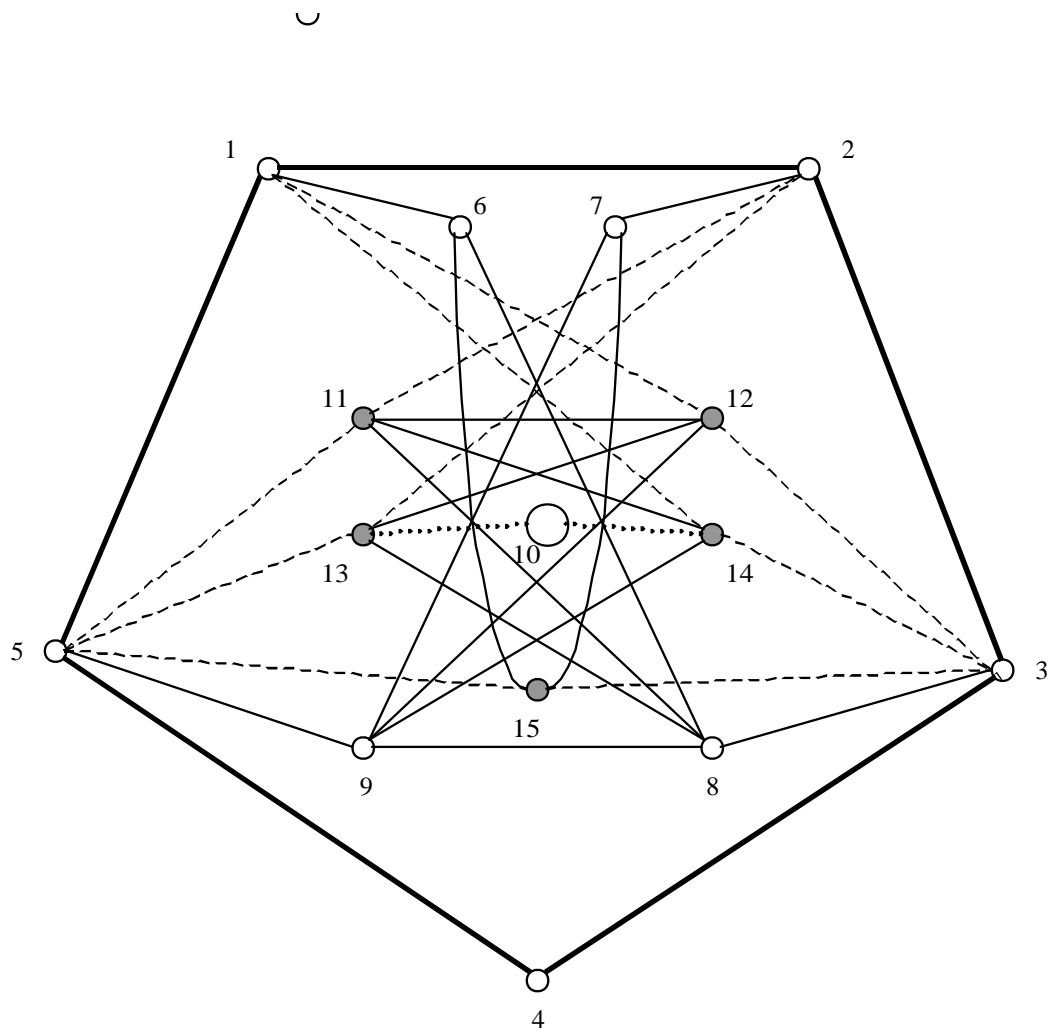


Figure 4: The graph H_3

6 The stable set polytope of $(P_6, \text{triangle})$ -free graphs

In this section let us describe the stable set polytope of $(P_6, \text{triangle})$ -free graphs (and more generally of (P_6, paw) -free graphs).

Remark 3. The stable set polytope of specific graphs has been computed by an adapted version of the software PORTA (available on line www.zib.de/Optimization/Software/Porta/), which Prof. Caterina De Simone (IASI-CNR, Rome) kindly sent me.

Theorem 7 *Let \mathcal{X} be the class of $(P_6, \text{triangle})$ -free graphs. Then $\mathcal{F}_P(\mathcal{X}) = \{K_2, C_5, G_1, \dots, G_{24}\}$, where G_1 be the graph drawn in Figure 5, and referring to the statement of Theorem 6: $G_2 = H_2$, $G_3 = H_2 - \{1\}$, $G_4 = H_2 - \{1, 2\}$, $G_5 = H_2 - \{1, 2, 3\}$, $G_6 = H_2 - \{1, 2, 4\}$, $G_7 = H_2 - \{1, 2, 3, 4\}$, $G_8 = H_2 - \{1, 2, 3, 12\}$, $G_9 = H_2 - \{1, 2, 3, 13\}$, $G_{10} = H_2 - \{1, 2, 3, 4, 5\}$, $G_{11} = H_2 - \{1, 2, 3, 4, 12\}$, $G_{12} = H_2 - \{1, 2, 3, 4, 5, 11\}$, $G_{13} = H_2 - \{1, 2, 3, 4, 5, 11, 14\}$, $G_{14} = H_3$, $G_{15} = H_3 - \{4\}$, $G_{16} = H_3 - \{15\}$, $G_{17} = H_3 - \{4, 12\}$, $G_{18} = H_3 - \{4, 11, 12\}$, $G_{19} = H_3 - \{4, 9, 12\}$, $G_{20} = H_3 - \{4, 5, 11, 12\}$, $G_{21} = H_3 - \{4, 10, 12, 13\}$, $G_{22} = H_3 - \{4, 7, 9, 11, 12\}$, $G_{23} = H_3 - \{4, 10, 11, 12, 13\}$, $G_{24} = H_3 - \{4, 10, 11, 12, 13, 14\}$.*

Proof. By Theorem 6, the elements of $\mathcal{F}_P(\mathcal{X})$ are subgraphs of H_1, H_2, H_3 . Then they can be detected by computing the stable set polytope of graphs H_1, H_2, H_3 , i.e., one has to detect the prime facet-inducing subgraphs of those three graphs. Then let us refer to Remark 3.

$\text{STAB}(H_1)$ provides the following facet-inducing subgraphs up to isomorphism: K_2, C_5, G_1 (they are the only $(P_6, \text{triangle})$ -free ones).

$\text{STAB}(H_2)$ provides the following facet-inducing subgraphs up to isomorphism: $K_2, C_5, G_1, \dots, G_{13}$.

$\text{STAB}(H_3)$ provides the following facet-inducing subgraphs up to isomorphism: $K_2, C_5, G_1, G_{14}, \dots, G_{24}$. \square

Then by Proposition 1, Observation 4, and Theorem 7 one obtains:

Theorem 8 *Let G be a $(P_6, \text{triangle})$ -free graph. Then a (minimal) linear system of $\text{STAB}(G)$ is given by:*

- (a) $-x_i \leq 0$ for every node i of G ;
- (b) $\{\Phi(H) : H \in \{K_2, C_5, G_1, \dots, G_{24}\}\}$, where graphs G_1, \dots, G_{24} are defined in the statement of Theorem 7. \square

Let us compare this result with that concerning the class, say \mathcal{X}' , of $(P_5, \text{triangle})$ -free graphs: as one can check from [7], or from Theorem 7 by ignoring those graphs containing a P_5 , one has $\mathcal{F}_P(\mathcal{X}') = \{K_2, C_5\}$; in particular, according to Observation 4, $(P_5, \text{triangle})$ -free graphs are t -perfect [15].

Then let us consider (P_6, paw) -free graphs. Let us report the following result of Olariu [20].

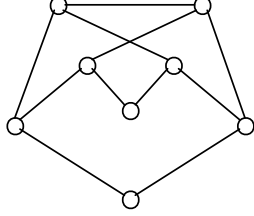


Figure 5: The graph G_1

Theorem 9 ([20]) *Every connected paw-free graph is either triangle-free or complete multipartite (that is it admits a partition into stable sets which have mutually a join).* \square

By combining Theorem 9 with the above one obtains the following corollaries.

Corollary 2 *Let \mathcal{Y} be the class of (P_6, paw) -free graphs. Then $\mathcal{F}_{\mathcal{P}}(\mathcal{Y}) = \{K_2, C_5, G_1, \dots, G_{24}\}$, where graphs G_1, \dots, G_{24} are defined in the statement of Theorem 7.* \square

Proof. Let G be a prime facet-inducing (P_6, claw) -free graph. Since every complete multipartite graph is perfect, and since every perfect prime facet-inducing graph is a K_2 , Theorem 9 implies that G is triangle-free. Then the corollary follows by Theorem 7. \square

Let F be a subgraph of a graph G . Let $C(F, G)$ the family of subgraphs H of G such that: (i) H contains F , and (ii) if H properly contains F , then $H - F$ is a clique and has a join to F . In particular $F \in C(F, G)$. An element H of $C(F, G)$ is *maximal* if no element of $C(F, G)$ properly contains H . Let $C^*(F, G)$ be the family of maximal elements of $C(F, G)$. For instance, $C^*(K_2, G)$ is the family of maximal cliques of G .

Corollary 3 *Let G be a (P_6, paw) -free graph. Then a (minimal) linear system of $\text{STAB}(G)$ is given by:*

- (a) $-x_i \leq 0$ for every node i of G ;
- (b) $\{\Phi(H) : H \in C^*(K_2, G) \cup C^*(C_5, G) \cup C^*(G_1, G) \cup \dots \cup C^*(G_{24}, G)\}$, where graphs G_1, \dots, G_{24} are defined in the statement of Theorem 7. \square

Proof. Let \mathcal{Y} be the class of (P_6, paw) -free graphs. By Corollary 1, $\mathcal{F}(\mathcal{Y}) = \mathcal{S}(\mathcal{F}_{\mathcal{P}}(\mathcal{Y})) \cap \mathcal{Y}$. The set $\mathcal{S}(\mathcal{F}_{\mathcal{P}}(\mathcal{Y})) \cap \mathcal{Y}$ depends on $\mathcal{F}_{\mathcal{P}}(\mathcal{Y})$, which is given by Corollary 2. Then any graph in $\mathcal{F}_{\mathcal{P}}(\mathcal{Y})$ can be (repeatedly) substituted for vertices of just one graph in $\mathcal{F}_{\mathcal{P}}(\mathcal{Y})$, namely of K_2 , since otherwise a paw arises. That is, $\mathcal{F}(\mathcal{Y})$ is given by $C(K_2, G) \cup C(C_5, G) \cup C(G_1, G) \cup \dots \cup C(G_{24}, G)$. Then for a minimal description of $\text{STAB}(G)$ one may just consider $C^*(K_2, G) \cup C^*(C_5, G) \cup C^*(G_1, G) \cup \dots \cup C^*(G_{24}, G)$. Then the corollary follows by Proposition 1. \square

The Maximum (Weight) Stable Set Problem and the Maximum (Weight) Clique Problem for $(P_6, \text{triangle})$ -free graphs can be solved in polynomial time (i.e., $O(n^2)$ time) [4]. That can be extended to (P_6, paw) -free graphs by Theorem 9. Then the following fact concerns the separation problem – see e.g. [15, 24, 25, 26]

Theorem 10 *The separation problem for $\text{STAB}(G)$ by facets can be solved in polynomial time when G belongs to the class of $(P_6, \text{triangle})$ -free graphs or more generally of (P_6, paw) -free graphs.*

Proof. Let G be a graph of n vertices. Let \mathbf{y} be any rational n -vector. Let us prove that when G is $(P_6, \text{triangle})$ -free and more in general (P_6, paw) -free there exists a polynomial time algorithm that either asserts that \mathbf{y} belongs to $\text{STAB}(G)$ or finds a facet-defining inequality of $\text{STAB}(G)$ violated by \mathbf{y} .

First assume that G is $(P_6, \text{triangle})$ -free. The nonnegativity constraints can be checked by substitution. So one may assume that the entries of \mathbf{y} are nonnegative. By Theorem 8, the remaining facet-defining inequalities of $\text{STAB}(G)$ are given by the set $\{\Phi(H) : H \in \{K_2, C_5, G_1, \dots, G_{24}\}\}$. Then, since the cardinality of $\{\Phi(H) : H \in \{K_2, C_5, G_1, \dots, G_{24}\}\}$ is bounded by a constant, such inequalities can be checked by substitution.

Then assume that G is (P_6, paw) -free. The nonnegativity constraints can be checked by substitution. So one may assume that the entries of \mathbf{y} are nonnegative. By Corollary 3, the remaining facet-defining inequalities of $\text{STAB}(G)$ are given by the set $\{\Phi(H) : H \in C^*(K_2, G) \cup C^*(C_5, G) \cup C^*(G_1, G) \cup \dots \cup C^*(G_{24}, G)\}$. The inequalities from $C^*(K_2, G)$ can be checked by solving the maximum weight clique problem in G (weighted by \mathbf{y}): as remarked above, this can be done in polynomial time. Then assume that \mathbf{y} satisfies also the inequalities from $C^*(K_2, G)$. Then every (not necessarily maximal) clique inequality is satisfied. This implies that, if \mathbf{y} violates an inequality over the remaining facet-defining inequalities of $\text{STAB}(G)$, then there exists a facet-defining inequality of $\text{STAB}(G[H])$ for $H \in T = \{C_5, G_1, \dots, G_{24}\}$ which is violated as well (that comes also from the structure of the elements of $C^*(H, G)$). As shown in Theorem 7, the facet-inducing subgraphs of a graph $H \in T$ are contained in $T \cup \{K_2\}$. Then the set of facet-defining inequalities of $\text{STAB}(G[H])$ for $H \in T$ is contained in $\{\Phi(H) : H \in \{K_2, C_5, G_1, \dots, G_{24}\}\}$. Then, since the cardinality of $\{\Phi(H) : H \in \{K_2, C_5, G_1, \dots, G_{24}\}\}$ is bounded by a constant, such inequalities can be checked by substitution. \square

7 A note on new facet-inducing graphs

In this section let point out some peculiarities of new facet-inducing graphs detected along this study with the help of a software according to Remark 3. Graphs H_1, H_2, H_3 , drawn

respectively in Figure 2, 3, 4, are (prime) facet-inducing. In particular let us list some peculiarities of graphs H_1 and H_2 .

Graph H_1 .

- (i) H_1 is facet-inducing;
- (ii) $H_1 - \{v\}$ is facet-inducing for every vertex v of H_1 ;
- (iii) $\text{STAB}(H_1 - \{16\})$ has 641 full facets, i.e., $|\Phi(H_1 - \{16\})| = 641$.

Graph H_2 .

- (iv) H_2 is facet-inducing;
- (v) $\text{STAB}(H_2)$ has 26617 facets;
- (vi) $H_2 - \{u\}$ is isomorphic to $H_2 - \{v\}$ for every pair of vertices u, v of H_2 ;
- (vii) $H_2 - \{u, v\}$ is isomorphic to $H_2 - \{u', v'\}$ for every pair of disjoint edges $uv, u'v'$ of H_2 ;
- (viii) $H_2 - \{v\}$ is facet-inducing for every vertex v of H_2 ;
- (ix) $H_2 - \{u, v\}$ is facet-inducing for every edge uv of H_2 ;
- (x) H_2 seems to enjoy further properties, such as regularity ...

Concerning (ii), (viii) and (ix), in a not expert and poor knowledge we ignore any other facet-inducing graph with those properties (apart from cliques); concerning (iii), similarly we ignore any other facet-inducing graph, of that low order, whose stable set polytope has a so large number of full facets; concerning (v), similarly we ignore any other facet-inducing graph, of that low order, whose stable set polytope has a so large number of facets.

Graphs H_1 and H_2 necessarily have a symmetric structure. In particular maybe one could expect that graphs H_1 and H_2 could lead to new families of (prime) facet-inducing graphs. To this end let us try to point out a symmetric representation for graphs H_1 and H_2 . Concerning graph H_1 , maybe Figure 2 seems to be enough in this sense since it shows a symmetry based on three C_5 's (one of which with a bigger out-degree) plus one external vertex, namely vertex 16. Concerning graph H_2 , also recalling that H_2 is regular, let us try to provide below a highly symmetric representation. Let us refer to Figure 6.

As a preliminary let us observe that, it seems that there is no ordering of vertices of graph H_2 , say v_1, \dots, v_{16} , such that each vertex v_i admits a list of neighborhoods formed by those vertices v_{i+j} which are at the same fixed distances (sum taken modulo 16).

A first representation of graph H_2 may be obtained by considering two copies, say A and B , of a graph of 8 vertices (Figure 6 up). The vertex-set of A (of B) can be uniquely partitioned into pairs, formed by vertices which have the maximum distance each other in A (in B). One of such a pair in A (in B) is distinguished in Figure 6 up by color gray. Then add edges in order to have a join from respectively each such a pair in A to its homologous pair in B .

A second representation of graph H_2 may be obtained by considering a C_4 , say C , of vertices a, b, c, d and of edges ab, bc, cd, da . Then expand each vertex of C into a

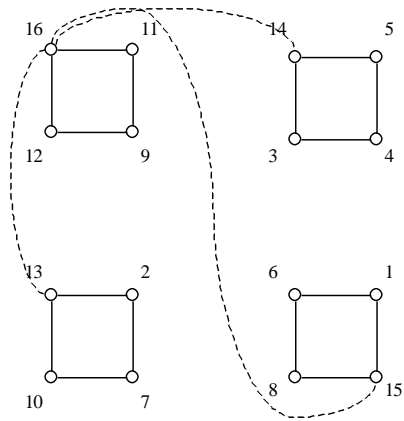
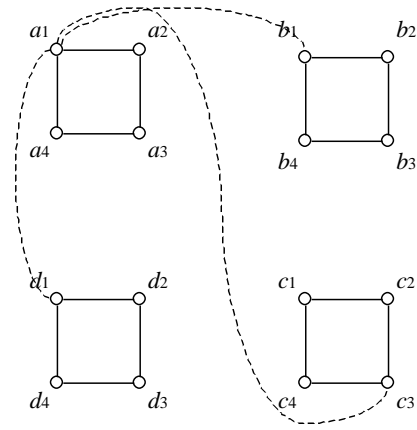
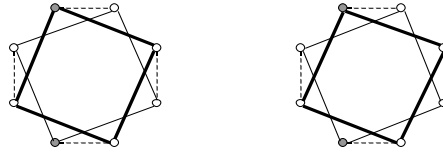


Figure 6: Representations of graph H_2

C_4 , that is for $x = a, b, c, d$, expand vertex x into a C_4 of vertices x_1, x_2, x_3, x_4 and edges $x_1x_2, x_2x_3, x_3x_4, x_4x_1$. Then add the following edges: for $i = 1, 2, 3, 4$, add edges $a_ib_i, b_ic_i, c_id_i, d_ia_i$, i.e., add edges between homologous vertices of the C_4 's when such C_4 's correspond to adjacent vertices of C , and add edges a_ic_{i+2}, b_id_{i+2} (sum taken modulo 4), i.e., add edges between opposite vertices of the C_4 's when such C_4 's correspond to nonadjacent vertices of C (Figure 6 middle). In particular the isomorphism from this representation to graph H_2 of Figure 6 is based on the following bijection between vertex-sets: from $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4$ to $16, 11, 9, 12, 14, 5, 4, 3, 6, 1, 15, 8, 13, 2, 7, 10$ (Figure 6 down).

Acknowledgement. I would like to thank Prof. Gianluca Amato, Prof. Marco Dall'Aglio, and Prof. Caterina De Simone for the help given to use the software Mathematica and the software mentioned in Remark 3. Then would like to thank Prof. Caterina De Simone for having improved the presentation of the proof of Proposition 1 of [19], i.e., of Proposition 2.

Then would like to thank for different reasons many persons which I met in these years of study, in Italy, Prof. Claudio Arbib for having introduced me to this study, for his support, and for having taught me many things, Prof. Giandomenico Boffi, Prof. Marco Dall'Aglio, Prof. Caterina De Simone, Prof. Paolo Nobili, Prof. Antonio Sassano, out of Italy, Prof. Andreas Brandstädt for many reasons detailed below, Prof. Hoàng-Oanh Le, Prof. Van Bang Le, Prof. Vadim V. Lozin for many cooperations so instructive to me, Prof. Haiko Müller, and generally other persons.

Then would like to thank in a special way Prof. Andreas Brandstädt for his support at a moment in which I had no position and no support, for having taught me many things, for many cooperations so instructive to me, and for having been a lighten benchmark under different aspects.

Finally would like humbly to dedicate this paper to (the memory of) my parents, namely, Giuseppe Mosca and Meris Ciufolini.

Just try to pray a lot and am able to do nothing without that.

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